Peter Mattock

Visible Maths

Using representations and structure to enhance mathematics teaching in schools

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This book is dedicated to:

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But mostly this book is dedicated to my loving partner Rowan. Her understanding and support throughout my career and throughout this project has been beyond what any man could hope for. She has put up with so much whilst I have worked to get to where I am in my career, and even more as I balanced leading a mathematics department with writing this book. Without her love this book would never have been written.

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Contents

Introduction

There is a great mathematics story that I was told in a lecture at university. It involves two donkeys and a fly. The problem goes that two donkeys are 100 metres apart and walking directly towards each other at 1 metre per second. A fly starts on the nose of the first donkey and buzzes between the noses of the two donkeys at 10 metres per second. The question is, how long before the fly is crushed between the two donkeys?

One of the ways to solve this problem is summing an infinite series (i.e. summing the terms of a sequence that continues forever). On its way to the second donkey the fly is travelling for $\frac{100}{11}$ seconds, then on the way back $\frac{900}{121}$ seconds, then another $\frac{8100}{1331}$ seconds, and so on. The *n*th term of the geometric series is given by $\frac{100}{11} \times (\frac{9}{11})^{n-1}$ and so the sum to infinity of the series is $\frac{100/11}{1-9/11} = \frac{100/11}{2/11} = \frac{100}{2} = 50$ seconds.

The other way to solve the problem is to ignore the fly completely. Each donkey is walking at 1 metre per second. This means that they will meet halfway at 50 metres. If they travel 50 metres at 1 metre per second it will take 50 seconds.

The story goes that a group of university students were told that a natural mathematician would automatically try to solve the problem using an infinite series and a natural physicist would solve it using the simpler approach. The problem therefore sorted mathematicians from physicists: if a student were able to solve it in a few seconds they were a physicist and if not they were a mathematician. The undergraduates were posing the problem to various students passing through the university library when the famous mathematician Leonhard Euler walked by. They presented the problem to Euler and were amazed when he answered the problem within a few seconds, as they had automatically expected him to begin considering the infinite series. When one of the students explained that a natural mathematician would have begun by forming the infinite series for the motion of the fly, Euler replied, 'But that is what I did …'

A very similar story exists about the eminent mathematician and computer scientist John von Neumann and trains, which makes me suspect that this is at best a parable about Euler and at worst a case of Chinese whispers. However, the point of the story is not to show how good at mathematics Euler (or von Neumann) was, but instead to show that sometimes in mathematics the way you think about the calculation or problem you are solving has a great impact on how simple the problem is or how much sense it makes. Only the best A level mathematics students would be able to form the infinite series necessary to solve the problem, whereas most early secondary school pupils would be able to work out the simpler solution.

The importance of having different ways to view even the most simple mathematics, in order to build up to more complicated ideas, cannot be overstated. Some ways of thinking about numbers make some truths self-evident, whilst simultaneously obscuring others. In the same way in physics that it is sometimes better to view elementary matter as particles and at other times as waves, so in mathematics it is sometimes better to view numbers as **discrete** and at other times as **continuous**, as counters or bars, as tallies or **vectors**. Crucially for teachers, being explicit about how we are thinking about numbers and operations, and encouraging pupils to think about them in different ways, can add real power to their learning.

Much has been made of the effectiveness of metacognition in raising the attainment of pupils. For example, John Hattie lists metacognitive strategies as having an effect size of 0.6 in the most recent list of factors influencing student achievement.[*] Ofsted also recognises the importance of the use of manipulatives and representations to support flexibility in pupil thinking. In their *Mathematics: Made to Measure* report from 2012, it is noted that schools should choose 'teaching approaches and activities that foster pupils' deeper understanding, including through the use of practical resources, [and] visual images'.[†] In *Improving Mathematics in Key Stages Two and Three*, the Education Endowment Foundation lists 'Use manipulatives and representations' as one of its key recommendations.[‡] It is therefore important that we give the pupils the tools they need in order to think about the mathematics they are working with in different ways.

The use of representations and structure is also an important part of teaching for mastery approaches. The National Centre for Excellence in the Teaching of Mathematics (NCETM) lists representation and structure as one of the 'Five Big Ideas' in teaching for mastery.[§] The NCETM make clear that using appropriate representations in lessons can help to expose the mathematical structure being taught, allowing pupils to make connections between and across different areas of maths. They also emphasise that the aim in using these representations is that pupils will eventually understand enough about the structure such that they do not need to rely on the representation any more. This is often summarised as employing a concrete-pictorial-abstract (or CPA) approach to teaching mathematics.

Recently re-popularised in the UK following the focus on teaching approaches imported from places such as Shanghai and Singapore, the CPA approach actually has at least some of its roots in the 1982 Cockcroft Report, which reviewed the teaching

* See https://www.visiblelearningplus.com/sites/default/files/250%20Influences.pdf.

† Ofsted, *Mathematics: Made to Measure* (May 2012). Ref: 110159. Available at: https://www.gov.uk/government/publications/mathematics-made-to-measure, p. 10.

‡ See P. Henderson, J. Hodgen, C. Foster and D. Kuchemann, *Improving Mathematics in Key Stages Two and Three: Guidance Report* (London: Education Endowment Foundation, 2017). Available at: https://educationendowmentfoundation.org.uk/public/files/Publications/Campaigns/Maths/KS2_KS3_Maths_Guidance_2017.pdf, pp. 10–13.

§ See https://www.ncetm.org.uk/resources/50042.

of maths in England and Wales.* The Cockcroft Report advocated (among many other things) the need to allow pupils the opportunity of practical exploration with concrete materials before moving towards abstract thinking.

There are several studies on the use of manipulatives across the age and ability range, with most showing that mathematics achievement is increased through the long-term use of concrete materials. The most comprehensive of these is Sowell's 'Effects of Manipulative Materials in Mathematics Instruction', a meta-analysis of 60 individual studies designed to determine the effectiveness of mathematics instruction with manipulative materials.† Those surveyed ranged in age from pre-school children to college-age adults who were studying a variety of mathematics topics. Sowell found that 'mathematics achievement is increased through the long-term use of concrete instructional materials and that students' attitudes toward mathematics are improved when they have instruction with concrete materials provided by teachers knowledgeable about their use'.‡

The aim of this book is to explore some of the different concrete materials available to teachers and pupils, ways of using these concrete and pictorial approaches to represent different types of numbers as discrete or continuous, how certain operations work when viewing numbers in these ways, and how these various representations can help to support the understanding of different concepts in mathematics. The book will look at the strengths of each representation, as well as the flaws, so that both primary and secondary school teachers of mathematics can make informed judgements about which representations will benefit their pupils. I will draw on my own experience of using the representations, as well as experiences shared by others, and appropriate research in order to support teachers in understanding how these representations can be implemented in the classroom.

How to use this book

I have often noticed that one of the difficulties pupils have in acquiring new mathematical understanding is that we introduce new ways of representing or thinking about mathematics at the same time as we try to teach a new mathematical concept or skill. I will take an alternative approach here, which is to explore all of the representations first and then, once they are secure, examine how more complicated calculations and concepts can be developed.

* W. H. Cockcroft (chair), *Mathematics Counts: Report of the Committee of Inquiry into the Teaching of Mathematics in Schools* [Cockcroft Report] (London: HMSO, 1982). Available at: http://www.educationengland.org.uk/documents/cockcroft/cockcroft1982.html.

† E. J. Sowell, Effects of Manipulative Materials in Mathematics Instruction, *Journal for Research in Mathematics Education*, 20(5) (1989), 498–505. Available at: http://www.jstor.org/stable/749423?read-now=1&seq=7#references_tab_contents.

‡ Sowell, Effects of Manipulative Materials in Mathematics Instruction, 498.

I wouldn't introduce all of these representations at once with pupils; instead I would introduce two or three. Importantly, though, I would ensure that pupils are comfortable with the representation before trying to use the representation to explore a new concept. This generally involves introducing the representation to pupils within a concept they are comfortable with, and modelling with them how the representation fits with what they already know. This then allows the teacher to develop the concept into something new, using the representation as a bridge.

As this book is aimed at teachers, Chapter 1 will set out all of the representations within the secure concept of whole numbers, and Chapter 2 will then extend these representations to include fractions and decimals. The basic operations of addition and subtraction of whole numbers will be introduced in Chapter 3, followed by multiplication and division of whole numbers in Chapter 4, and powers and roots of whole numbers in Chapter 5. Chapter 6 then explores these ideas as applied to fractions and decimals. Chapter 7 examines the use of representations to illustrate the fundamental laws of arithmetic, and then in Chapter 8 we look at how these combine to define the correct order of operations in calculations involving multiple operations. Chapter 9 covers the concepts of accuracy, including **rounding**, significant figures and bounds, before we move on to **irrational numbers** in Chapter 10.

Chapter 11 sees the introduction of different representations applied to algebra, after which we progress to manipulating algebraic expressions by simplifying expressions (Chapter 12), multiplying expressions (Chapter 13) and expanding and factorising expressions (Chapter 14). In Chapter 15 we look at how representations can support with illustrating the solutions of **equations**, and then Chapter 16 examines some particular algebraic manipulations not covered in Chapters 12 to 14 – in particular, the difference of two squares and completing the square. Finally, Chapter 17 seeks to answer some of the questions about the use of representations in the classroom that may arise from the reading of the book.

You will notice while reading the book that some key mathematical terms are presented in **bold** – for your convenience these terms are defined in a glossary, found at the back of the book.

The fact that the book spans almost the complete breadth of primary and secondary school mathematics might make some question the usefulness of covering everything in one text. One reason I have chosen to do so is that I feel it is important that teachers understand not just the stage they are teaching, but also how this builds on what has been taught before and how this is built on in the stages after. This ensures that teachers see how what they are teaching fits into the wider pupil journey, and can support pupils no matter where they are along the way. Pupils will enter and leave stages of schooling at many different points, and just because we might teach in a secondary school doesn't mean we won't need to support pupils who haven't secured concepts from primary school, or similarly that teachers in primary schools won't need to provide depth in a topic by allowing pupils to explore a concept to a point that would normally be taught in secondary school. In this, all-through (3–18) schools have an advantage as they can design their curriculum to build all the way through the school. Those working in separate primary and secondary schools, or other school models, must use strong transition links to make this happen. So, for primary school teachers, this book showcases the mathematics you will teach and show you how it extends into secondary school. For secondary teachers, this book will provide some insight into approaches that might be used in feeder primaries and how you can develop them in secondary school.

I hope this book will support teachers in choosing suitable representations for use in their classrooms by making them much more secure in their own understanding of the strengths and weaknesses of each representation, but also, importantly, of how the representations highlight different interpretations of the concepts we explore with pupils. Some of the examples in the book will be suitable for direct use with pupils in the classroom, whilst some will be of more benefit to teachers in developing their own understanding. Pupils will very often need more than the one or two examples illustrated at each stage; in many cases, they will need to experience careful modelling with multiple examples as well as have the opportunity to explore concepts with the different manipulatives and representations provided. Only in this way will pupils eventually move beyond the representations.

The true aim of this book is for teachers to feel sufficiently confident in the use of the representations that they can explain enough about the underlying structures of the different concepts so that pupils no longer need to rely on the representations to see these structures. This is an important end goal for teachers to keep in mind – pupils should be aiming to move beyond the representation. Representations are tools that provide a window into the underlying structure of a concept. They are a window that pupils can keep coming back to look into, but they are not a window they should continually have to stare through. There is a danger that representations become another procedure that pupils have to remember and apply without understanding; this must be avoided at all costs if pupils are going to work towards mastery of mathematical concepts. This is why multiple representations are used for each concept, and why the literature makes clear the need for multiple representations to ensure pupils have a range of ways of thinking about concepts.

Chapter 1

Different representations of whole numbers

Many people believe that counting was the earliest mathematical concept to emerge. Whilst counting can be traced back several thousands of years, the first mathematical idea was actually the one-to-one relationship – relating a number of objects with an equal number of different objects. According to Kris Boulton, ancient shepherds would allow their sheep out to graze during the day, and for every sheep that went out they would put a stone into a pot. At the end of the day, the shepherds would bring the sheep back into the pens to keep them safe from predators. As each animal returned, the shepherds would remove a stone from the pot. When the pot was empty, all the sheep were safely back.* Interestingly, it seems that no concept was required for how many sheep there were, just that the number of sheep was equal to the number of stones.

The earliest example of counting itself is thought to be the Ishango bone, which bears scratch marks grouped in 60s. Discovered in Africa in 1960, the bone is believed to be more than 11,000 years old and is seen by many as the earliest example of a mathematical structure.† There is still not complete consensus about what these scratch marks represent, but one theory is that they are related to some form of lunar calendar. This would make sense given the prevalence of the number 60 in ancient time-keeping; indeed, our own 60-minute hour and 60-second minute can be traced back to the ancient Babylonians and their **base** 60 number system.

Both of these approaches treat numbers as discrete objects: you have one sheep, two sheep or three sheep (even though the shepherds weren't actually counting them); you have one scratch, two scratches or three scratches. These earliest occurrences of numerical relationships are still relatable to a mathematical representation that we use today to show and track discrete values – tallying.

* Kris recounted this story in his presentation 'The Stories of Mathematics – Part 1' at the Complete Mathematics Conference 5, Sheffield, 26 September 2015.

† See http://www.math.buffalo.edu/mad/Ancient-Africa/ishango.html.

Tallying

Tallying is probably one of the most basic representations of discrete number. In the English national curriculum, tally charts are introduced in Year 2 (age 6–7), although they could be used as a pictorial representation of number in Year 1 or earlier. Children are often taught to count before entering any statutory stage of education, and in the Early Years Foundation Stage statutory framework it is required for pupils to 'improve their skills in counting, understanding and using numbers, [and] calculating simple addition and subtraction problems'.*

The use of one mark per item to count harks back all the way to the earliest one-to-one relationships, and is a representation of number that nearly all mathematics students can grasp. The basic tenet of the representation is that a vertical line is used to represent a discrete value (normally 1) and these are grouped together in 5s or 10s when counting large numbers:

| = 1 || = 2 ||| = 3 |||| = 4   ~~||||~~ = 5   ~~||||~~ | = 6

It is very unusual for a tally mark to stand for anything other than 1, although theoretically it is possible (e.g. using pictograms to represent data takes advantage of this idea to represent large numbers). Tallying is severely deficient as a representation of number for anything other than the counting of a small discrete number of objects. Representing negative numbers is also problematic, to the point where no one would really consider using it. However, it is a valid representation of discrete number that can support young children to master counting and create a semi-permanent record, so its value should not be underestimated. Indeed, much medieval accounting was done using tallying – marks representing the value of goods or items traded or borrowed would be carved onto a piece of wood using large tally marks. The wood would then be split along its length so the notches appeared on each half. This provided both parties with a record of the trade that couldn't be altered, and only those two pieces of wood could fit together to confirm they were records of the same trade. Students of mathematics should definitely be aware of tally marks as a representation for counting small discrete values, and possibly some of the history around them, but they should also be aware of the limitations of this representation in moving mathematics forward.

It wasn't until the emergence of complex civilisations that more sophisticated views of numbers developed. As early as 4000 BC, the ancient Sumerians lived in cities, some of which may have had up to 80,000 residents. This required proper

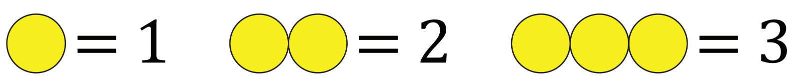
* Department for Education, *Statutory Framework for the Early Years Foundation Stage: Setting the Standards for Learning, Development and Care for Children from Birth to Five* (March 2017). Available at: https://www.gov.uk/government/publications/early-years-foundation-stage-framework--2, p. 8.

administration and consequently more sophisticated mathematics. Taxes needed to be collected and recorded, resources counted and measured, wealth calculated and compared. It is here that we see the birth of one of the more versatile discrete number representations – tokens, or counters.

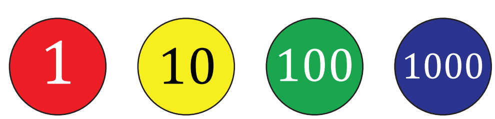
Counters

Counters certainly have many advantages as a representation of discrete numbers when compared to tallying. It is much easier to assign different values to counters (think of the number of different value coins that have existed in various world currencies) and so take relationships beyond the one-to-one relationships that were a hallmark of very early mathematics and counting systems. The fact that counters can be removed, as well as added to, allows for the development of arithmetic, which was crucial in developing the mathematics required to manage the complex financial calculations needed to administrate a city.

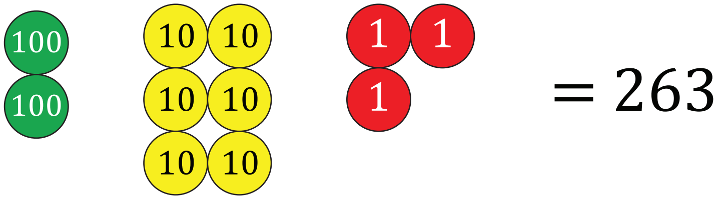
In the mathematics classroom, counters can be used in a variety of ways to support pupils' understanding of different types of numbers. At a simple level, counters can be used to represent positive **integers** in the same way that tallies do:



However, the versatility of counters more readily allows for them to hold different values, either by using different colour counters or ones that can be written on. For example, place value counters can be used to support an understanding of large numbers:

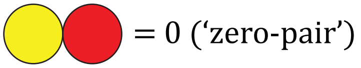


This allows large numbers to be represented without needing thousands of counters – for example:



In addition to representing larger numbers, counters also have an advantage over tallies in that they can simultaneously represent positive and negative numbers, which is done using either two different colours or, if available, double-sided counters:

The ability to use counters to simultaneously represent positive and negative numbers means that counters are an excellent way to develop directed number arithmetic. Crucial to this is the understanding that a ‘1’ counter plus a ‘-1’ counter results in 0. Indeed, a pair of these together (like the pair below) are often called a zero-pair.



Both tallies and counters have one crucial drawback, however: they only represent numbers as discrete quantities. In both cases it isn’t clear that there are numbers between 1 and 2, and whilst it is possible to represent fractions and decimals using counters once these concepts are well defined, it is very difficult to introduce the idea of either fractions or decimal numbers using solely counters or tallies. The first representation that begins to show numbers as both discrete and continuous is one very familiar representation – the number line.

There's more to maths than finding the right answers – what's much more important is understanding where they come from

In *Visible Maths* Peter Mattock builds on this idea and explores, in colourful detail, a variety of visual tools and techniques that can be used in the classroom to illustrate key concepts and deepen pupils' understanding of mathematical operations.

Covering vectors, number lines, algebra tiles, ordered-pair graphs and many other representations, *Visible Maths* equips teachers with the confidence and practical know-how to take their pupils' learning to the next level.

Suitable for teachers of maths in primary and secondary school settings

An ideal guide for those educators who want to help their pupils 'see' and 'feel' the mathematics.

Steve Lomax, mathematics adviser and national teaching for mastery lead

Visible Maths is a thoughtful, careful and thorough exploration of some of the most useful visual models we can use to teach mathematics.

Lucy Rycroft-Smith, Research and Framework Design, Cambridge Mathematics

An invaluable resource to refer to again and again, this book deserves a place on the shelf in every school's maths department.

Jonathan Hall, Lead Maths Practitioner, Leeds City Academy and creator of mathsbot.com

Visible Maths provides a practical guide to using representations and manipulatives in the classroom, demonstrating how we can offer pupils coherence in the representations we choose to use, irrespective of the complexity of the topic we are studying.

Emma McCrea, teacher trainer and author of *Making Every Maths Lesson Count*

I recommend *Visible Maths* to all those who constantly consider different ways of supporting their teaching and their pupils' learning.

Geoff Wake, Professor of Mathematics Education and Convenor of the Centre for Research in Mathematics Education, University of Nottingham

Peter Mattock has been teaching maths for over a decade. He is a specialist leader of education (SLE) and an accredited secondary maths professional development lead, who regularly presents at conferences across the country. Peter also develops teaching for mastery in the secondary school classroom, having been part of the first cohort of specialists trained in mastery approaches by the National Centre for Excellence in the Teaching of Mathematics (NCETM). **@MrMattock**

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